

REMARKS ON LEMPERT FUNCTIONS OF BALANCED DOMAINS

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ABSTRACT. This note should clarify how the behavior of certain invariant objects reflects the geometric convexity of balanced domains.

1. INTRODUCTION AND RESULTS

By \mathbb{D} we denote the unit disc in \mathbb{C} . Let D be a domain in \mathbb{C}^n . Recall first the definitions of the Carathéodory pseudodistance and the Lempert function of D :

$$c_D(z, w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D, \mathbb{D}) : f(z) = 0\},$$

$$\tilde{k}_D(z, w) = \inf\{\tanh^{-1}|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\}.$$

The Kobayashi pseudodistance k_D can be defined as the largest pseudodistance below of \tilde{k}_D . Note that if $k_D^{(m)}$ denotes the m -th Lempert function of D , $m \in \mathbb{N}$, that is,

$$k_D^{(m)}(z, w) = \inf\left\{\sum_{j=1}^m \tilde{k}_D(z_{j-1}, z_j) : z_0, \dots, z_m \in D, z_0 = z, z_m = w\right\},$$

then

$$k_D(z, w) = \inf_m k_D^{(m)}(z, w).$$

If l_D is any one of the introduced functions from above, we set $l_D^* = \tanh l_D$.

Recall that D is said to be balanced if $\lambda z \in D$ for any $\lambda \in \overline{\mathbb{D}}$ and any $z \in D$. Denoting by h_D the Minkowski function of D , i.e.,

$$h_D(z) = \inf\{t > 0 : z/t \in D\}, \quad z \in \mathbb{C}^n,$$

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then

$$D = D_h = \{z \in \mathbb{C}^n : h_D(z) < 1\}.$$

We point out that D is pseudoconvex if and only if $\log h_D$ is a plurisubharmonic function. Set $\widehat{h}_D = h_{\widehat{D}}$, where \widehat{D} is the convex hull of D .

Let us summarize some well-known facts about relations between h_D , \widehat{h}_D , and the functions from above, where one of their arguments is the origin.

Proposition 1. (cf. [1]) *Let $D \subset \mathbb{C}^n$ be a balanced domain and $a \in \mathbb{D}$. Then*

- (i) $\widehat{h}_D \leq c_D^*(0, \cdot) \leq \tilde{k}_D^*(0, \cdot) \leq h_D$;
 - (ii) $k_D^*(0, a) = h_D(a) \iff h_D(a) = \widehat{h}_D(a)$.
- If, in addition, D is pseudoconvex, then*

$$\tilde{k}_D^*(0, \cdot) = h_D.$$

Proposition 1 shows that at a point $a \in D$ the value of $k_D(0, \cdot)$ is maximal if and only if D is "convex in the direction of a ", i.e., $h_D(a) = \widehat{h}_D(a)$. In fact, more is true as the following result shows.*

Proposition 2. *Let $D \subset \mathbb{C}^n$ be a balanced domain and $a \in D$. Then*

$$(k_D^{(3)}(0, a))^* = h_D(a) \iff h_D(a) = \widehat{h}_D(a).$$

Remark. We do not know if the number 3 can be replaced by 2.

Conversely, one may ask whether the fact that $l_D(0, a)$ (l_D as above) is "minimal" (i.e., $l_D(0, a) = h_G(a)$ for some domain G containing D) implies also some convexity property. Here we start with the following result.

Proposition 3. *Let $D \subset \mathbb{C}^n$ be a bounded balanced domain and $G \subset \mathbb{C}^n$ be a pseudoconvex balanced domain with $G \supset D$. Assume that h_D is continuous at some $a \in D$ and \overline{G} contains no nontrivial analytic discs through $a/h_G(a)$, $h_G(a) \neq 0$. Then*

$$\tilde{k}_D^*(0, a) = h_G(a) \iff h_D(a) = h_G(a).$$

Remarks. (a) Since the envelope of holomorphy $\mathcal{E}(D)$ of a balanced domain D is balanced (see [2], Remark 3.1.2(b)), one may apply the above result for D and $\mathcal{E}(D)$.

(b) If h_G is continuous near a and ∂G contains no nontrivial discs through $a/h_G(a)$, then the maximum principle implies that \overline{G} contains no nontrivial analytic discs through $a/h_G(a)$, too.

*The proofs of the following propositions and examples will be given in Section 2.

(c) In light of Proposition 3, it is natural to ask whether there is a non-pseudoconvex balanced domain D such that $h_D = \tilde{k}_D^*(0, \cdot)$ on D . The authors do not know the answer.

The following examples show that the assumptions about continuity of h_D at a and discs in Proposition 3 are essential.

Example 4. *If $D = \mathbb{D}^2 \setminus \{(t, t) : |t| \geq 1/2\}$, $d = (t, t)$, $|t| < 1/2$, then*

$$h_D(d) = 2|t| \quad \text{but} \quad \tilde{k}_D^*(0, d) = |t|.$$

On the other hand, $\overline{\mathbb{D}^2}$ contains no nontrivial analytic discs through any point of $\partial\mathbb{D} \times \partial\mathbb{D}$.

Moreover, the following example gives a balanced Reinhardt domain D such that $G = \mathcal{E}(D) = \widehat{D}$ has nontrivial analytic discs in its boundary for which Proposition 3 fails to hold. Note that in this case h_D and h_G are continuous functions.

Example 5. *Let $0 < a < 1$ and*

$$D = \{z \in \mathbb{D}^2 : |z_2|^2 - a^2 < 2(1 - a^2)|z_1|\}.$$

Then D is a balanced Reinhardt domain and $\mathcal{E}(D) = \mathbb{D}^2$ (cf. [2]). On the other hand, if $c = (0, d)$, $|d| < a$, then

$$h_D(c) = \frac{|d|}{a} > \tilde{k}_D^*(0, c) = |d| = \widehat{h}_D(c).$$

For a balanced domain D and $a \in D$ set

$$\mathbb{D}_{D,a} := \{\lambda a : |\lambda| h_D(a) < 1\}.$$

Assuming minimality along the whole slice $\mathbb{C}a \cap D = \mathbb{D}_{D,a}$ for some $a \in D$ we have

Proposition 6. *Let $D \subset \mathbb{C}^n$ be a taut balanced domain, $a \in D$ and $m \in \mathbb{N}$. Then*

$$(k_D^{(m)}(0, \tilde{a}))^* = \widehat{h}_D(\tilde{a}), \quad \tilde{a} \in \mathbb{D}_{D,a} \iff h_D(a) = \widehat{h}_D(a).$$

Recall that a domain $D \subset \mathbb{C}^n$ is said to be taut if $\mathcal{O}(\mathbb{D}, D)$ is a normal family. Note that a balanced domain D is taut if and only if h_D is a continuous plurisubharmonic function and $(h_D)^{-1}(0) = 0$. Weakening the continuity assumption for h_D the following statement remains true.

Proposition 7. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex balanced domain, $a \in D$ and $m \in \mathbb{N}$. Assume that h_D is continuous at a and ∂D contains no nontrivial analytic discs through $a/h_D(a)$. Then*

$$(k_D^{(m)}(0, \tilde{a}))^* = \widehat{h}_D(\tilde{a}), \quad \tilde{a} \in \mathbb{D}_{D,a} \iff h_D(a) = \widehat{h}_D(a).$$

Remark. We do not know if the condition about discs is superfluous or not. On the other hand, continuity and pseudoconvexity are essential as Examples 4 and 5 have shown (see also Example 9 below).

Recall that a boundary point b of a domain D in \mathbb{C}^n is said to be a *local weak barrier point* if there are a neighborhood U of b and a negative plurisubharmonic function u on $D \cap U$ such that $\lim_{D \ni z \rightarrow b} u(z) = 0$.

Proposition 7 is a consequence of the following

Proposition 8. *Let b be a local weak barrier point of a bounded domain $D \subset \mathbb{C}^n$. If ∂D contains no nontrivial analytic discs through b , then*

$$\lim_{w \rightarrow b} k_D^{(m)}(z, w) = \infty, \quad z \in D, \quad m \in \mathbb{N}.$$

We point out that the assumption about tautness is essential in Proposition 6 as the following examples show.

Example 9. *The unit ball \mathbb{B} in \mathbb{C}^2 contains a proper non-taut pseudoconvex balanced domain D with $\widehat{D} = \mathbb{B}$ such that*

$$(k_D^{(2)}(0, \cdot))^* = \|\cdot\|.$$

Example 10. *There is an unbounded pseudoconvex balanced Reinhardt domain D in \mathbb{C}^2 with continuous Minkowski function and a point $a \in D$ such that $\widehat{h}_D(a) > 0$, ∂D contains a nontrivial analytic disc through $a/h_D(a)$,*

$$(k_D^{(2)}(0, \lambda a))^* = |\lambda| \widehat{h}_D(a), \quad |\lambda| \leq 1,$$

but even

$$c_D^*(0, \lambda a) > |\lambda| \widehat{h}_D(a), \quad 1 < |\lambda| < 1/h(a).$$

Remark. Despite of these examples, we do not know any example of a taut balanced domain D such that $(k_D^{(m)}(0, a))^* = \widehat{h}_D(a)$ for some $m \in \mathbb{N}$ and some $a \in D$, but $h_D(a) > \widehat{h}_D(a)$.

2. PROOFS

Proof of Proposition 2. We have only to prove that

$$(k_D^{(3)}(0, a))^* = h_D(a) \Rightarrow h_D(a) = \widehat{h}_D(a).$$

First, we shall show that

$$(1) \quad (k_D^{(2)}(0, \lambda a))^* = |\lambda| h_D(a), \quad \lambda \in \mathbb{D}.$$

We may assume that $h_D(a) \neq 0$. Taking the disc $\mathbb{D} \ni t \rightarrow ta/h_D(a)$ as a competitor for $\tilde{k}_D(\lambda a, a)$ gives

$$\tilde{k}_D(\lambda a, a) \leq p(h_D(\lambda a), h_D(a)),$$

where p denotes the Poincaré distance. This and the inequality

$$p(0, h_D(a)) = k_D^{(3)}(0, a) \leq k_D^{(2)}(0, \lambda a) + \tilde{k}_D(\lambda a, a)$$

imply that

$$p(0, |\lambda| h_D(a)) = p(0, h_D(a)) - p(|\lambda| h_D(a), h_D(a)) \leq k_D^{(2)}(0, \lambda a).$$

So

$$(k_D^{(2)}(0, \lambda a))^* \geq |\lambda| h_D(a)$$

and the opposite inequality always holds.

It follows from (1) that

$$\lim_{\lambda \rightarrow 0} \frac{k_D^{(2)}(0, \lambda a)}{|\lambda|} = h_D(a).$$

On the other hand, by Proposition 2 in [4], this limit does not exceed

$$\kappa_D^{(2)}(0; a) := \inf\{\kappa_D(0; a_1) + \kappa_D(0; a_2) : a_1 + a_2 = a\},$$

where κ_D denotes the Kobayashi–Royden pseudometric of D . Since $\kappa_D(0; \cdot) \leq h_D$ (cf. [1]), we conclude that

$$h_D(a) \leq h_D(a_1) + h_D(a_2) \text{ if } a_1 + a_2 = a,$$

which means that $h_D(a) = \widehat{h}_D(a)$. □

Proof of Proposition 3. We have only to prove that

$$\tilde{k}_D^*(0, a) = h_G(a) \Rightarrow h_D(a) \leq h_G(a).$$

Let $(\varphi_j) \subset \mathcal{O}(\mathbb{D}, D)$ and $\alpha_j \rightarrow h_G(a)$ be such that $\varphi_j(0) = 0$ and $\varphi_j(\alpha_j) = a$. Writing φ_j in the form $\varphi_j(\lambda) = \lambda \psi_j(\lambda)$, $\psi_j \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$, it follows by the maximum principle that $h_G \circ \psi_j \leq 1$ and hence $\psi_j \in \mathcal{O}(\mathbb{D}, \overline{G})$. Since D is bounded, we may assume that $\varphi_j \rightarrow \varphi \in \mathcal{O}(\mathbb{D}, \overline{D})$ and then $\psi_j \rightarrow \psi \in \mathcal{O}(\mathbb{D}, \overline{G})$. On the other hand, since \overline{G} contains no nontrivial analytic discs through $\psi(h_G(a)) = b$, where $b = a/h_G(a)$, it follows that $\psi(\mathbb{D}) = b$. Using that h_D is continuous at b , we get that

$$1 > h_D(\varphi_j(\lambda)) \rightarrow |\lambda| h_D(b), \quad \lambda \in \mathbb{D}.$$

Letting $\lambda \rightarrow 1$ leads to $h_D(b) \leq 1$ which is the desired inequality. □

Proof of Example 4. We have only to prove that

$$\tilde{k}_D^*(0, d) \leq |t|.$$

For any $r \in (|t|, 1)$ we may choose $\alpha \in \mathbb{D}$ such that $t = \varphi(t/r)$, where $\varphi(\lambda) = \lambda \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}$. Then $\psi = (\text{rid}, \varphi) \in \mathcal{O}(\mathbb{D}, D)$ is a competitor for $\tilde{k}_D^*(0, d)$ which shows that $\tilde{k}_D^*(0, d) \leq |t|/r$. It remains to let $r \rightarrow 1$. □

Remarks. (a) Let D be the domain from Example 4. Note that even

$$\tilde{k}_D(0, \cdot) = \tilde{k}_{\mathbb{D}^2}(0, \cdot).$$

It is enough to prove that $\tilde{k}_D(0, a) \leq |a_1|$ for $a = (a_1, a_2) \in D$, $a_1 \neq a_2$, $|a_1| \geq |a_2|$. For this, take the discs $\psi(\lambda) = (\lambda, \lambda a_2/a_1)$ as a competitor for $\tilde{k}_D(0, a)$.

On the other hand, if $a_1 = (0, b)$ and $a_2 = (b, 0)$, $b \in \mathbb{D}$, then

$$\tilde{k}_D(a_1, a_2) = \tilde{k}_{\mathbb{D}^2}(a_1, a_2) \iff |b| \leq 4/5.$$

Indeed, using the Möbius transformation $\psi_b(\lambda) = \frac{\lambda-b}{1-\bar{b}\lambda}$, we get that

$$\tilde{k}_D(a_1, a_2) = \tilde{k}_{D_b}(0, a),$$

where $a = (b, -b)$ and $D_b = \mathbb{D}^2 \setminus \{(\psi_b(\lambda), \lambda) : 1/2 \leq |\lambda| < 1\}$.

For $|b| < 4/5$, it is easy to check that $\varphi = (\text{id}, -\text{id}) \in \mathcal{O}(\mathbb{D}, D_b)$. This implies that $\tilde{k}_{D_b}^*(0, a) \leq |b|$ and hence $\tilde{k}_D(a_1, a_2) = \tilde{k}_{\mathbb{D}^2}(a_1, a_2)$.

To get the same for $|b| = 4/5$, it is enough to take $r\varphi$, $r \in (0, 1)$, as a competitor for $\tilde{k}_{D_b}^*(0, a)$ and then to let $r \rightarrow 1$.

Assume now that $|b| > 4/5$ and $\tilde{k}_D(a_1, a_2) = \tilde{k}_{\mathbb{D}^2}(a_1, a_2)$. Then we may find discs $\varphi_j \in \mathcal{O}(\mathbb{D}, D_b)$ such that $\varphi_j(0) = 0$ and $\varphi_j(\alpha_j) = a$, where $\alpha_j \rightarrow b$. It follows by the Schwarz-Pick lemma that $\varphi_j \rightarrow \varphi$. On the other hand, $\varphi(\mathbb{D}) \cap \{(\psi_b(\lambda), \lambda) : 1/2 < |\lambda| < 1\}$ is a singleton which contradicts to Hurwitz's theorem.

(b) Note that (a) shows that Theorem 3.4.2 in [1] is in some sense sharp. On the other hand, this theorem implies that, if $D_n = \mathbb{D}^n \setminus \{(t, \dots, t) : |t| \geq 1/2\}$, $n \geq 3$, then

$$\tilde{k}_{D_n} = \tilde{k}_{\mathbb{D}^n}.$$

Proof of Example 5. We have only to prove that $\tilde{k}_D^*(0, c) \leq d$, $d \in (0, a)$. For this, it is enough to show that $\varphi = (\psi, \text{id}) \in \mathcal{O}(\mathbb{D}, D)$, where $\psi(\lambda) = \lambda \frac{\lambda-d}{1-d\lambda}$. Since $|\psi(\lambda)| \geq x \frac{x-d}{1-dx}$ for $x = |\lambda|$, we have to check that

$$(x^2 - a^2)(1 - dx) < 2(1 - a^2)x(x - d), \text{ i.e.,}$$

$$dx^3 + (1 - 2a^2)x^2 - d(2 - a^2)x + a^2 > 0.$$

Note that this inequality is true for $x = 0$. Using that $x \in (0, 1)$ and $d \in (0, a)$, it suffices to prove that

$$ax^3 + (1 - 2a^2)x^2 - a(2 - a^2)x + a^2 \geq 0$$

which is equivalent to the obvious inequality $(x - a)^2(ax + 1) \geq 0$. \square

Proof of Proposition 6. It is enough to show that if $\mathbb{D}_{D,a} \ni a_k \rightarrow a/h_D(a)$, $a \neq 0$, and $(k_D^{(m)}(0, \tilde{a}_k))^* = \widehat{h}_D(\tilde{a}_k)$, then $h_D(a) = \widehat{h}_D(a)$. For this, recall that if D is a taut domain, then (cf. [1], Proposition 3.2.1)

$$\lim_{w \rightarrow \partial D} k_D^{(m)}(z, w) = \infty, \quad w \in D.$$

In our case this implies that $\widehat{h}_D(\tilde{a}_k) \rightarrow 1$ and hence $\widehat{h}_D(a/h(a)) = 1$. \square

Proof of Proposition 8. We shall proceed by induction on m . For this, we shall need the following.

Lemma 11. *Under the assumptions of Proposition 8, for any $(\varphi_k) \subset \mathcal{O}(\mathbb{D}, D)$ with $\varphi_k(0) \rightarrow b$ one has that $\varphi_k \rightarrow b$ locally uniformly on \mathbb{D} .*

Assuming Lemma 11 easily implies that Proposition 8 is true for $m = 1$. Suppose that this statement is true for some $m - 1 \in \mathbb{N}$ but false for m . Then we may find $z \in D$ and sequences $(z_{j,k})_k \subset D$, $0 \leq j \leq m$, such that $z_{0,k} = z$, $z_{m,k} \rightarrow b$ and

$$\sup_k \sum_{j=1}^m \tilde{k}_D(z_{j-1,k}, z_{j,k}) < \infty.$$

In virtue of our induction hypothesis one has that $z_{m-1,k} \not\rightarrow b$. Passing to a subsequence, we may assume that $z_{m-1,k} \rightarrow a \in \overline{D}$, $a \neq b$ and $(\tilde{k}_D(z_{m-1,k}, z_{m,k}))^* \rightarrow r < 1$. Then there are $\varphi_k \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi_k(0) = z_{m,k}$, $\varphi_k(r_k) = z_{m-1,k}$ and $r_k \rightarrow r$ which contradicts Lemma 11. \square

Proof of Lemma 11. Since D is bounded, it is easily seen that for any neighborhood U of b there is another neighborhood $V \subset U$ of b and a number $s \in (0, 1]$ such that, if $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) \in V$, then $\varphi(s\mathbb{D}) \subset D \cap U$.

Now we choose U such that there is a negative plurisubharmonic function u on $D \cap U$ with $\lim_{D \ni z \rightarrow b} u(z) = 0$. Applying Proposition 3.4 in [3] implies that b is a t -point for $D \cap D$ which means that

(*) $(\varphi_j|_{s\mathbb{D}})$ is compactly divergent w.r.t. $D \cap U$.

Assume now that (φ_j) does not converge to b . Passing to a subsequence, we may assume that $\varphi_j \rightarrow \varphi \in \mathcal{O}(\mathbb{D}, \overline{D})$ and $\varphi(\mathbb{D}) \neq \{b\}$. It follows by (*) that $\varphi(s\mathbb{D}) \subset \partial(D \cap U)$. Since $\varphi(0) = b \in \partial D \cap U$, we may find $s' \in (0, s]$ such that $\varphi(s'\mathbb{D}) \subset \partial D$. Using that ∂D contains no nontrivial analytic discs through b , we get that $\varphi(s'\mathbb{D}) = \{b\}$ which contradicts the identity principle. \square

Proof of Example 9. J. Siciak (cf. [1], Example 3.1.12) constructed a plurisubharmonic function $\psi : \mathbb{C}^2 \rightarrow [0, \infty)$ such that $\psi(\lambda z) = |\lambda|\psi(z)$ ($\lambda \in \mathbb{C}, z \in \mathbb{C}^n$), $\psi \not\equiv 0$, but $\psi = 0$ on a dense subset S of \mathbb{C}^n . Set

$$D = \{z \in \mathbb{C}^2 : \|z\| + \psi(z) < 1\}.$$

For any $a \in D$ we may choose a sequence $S \cap D \supset (z_k) \rightarrow a$. Then

$$\tanh^{-1} \|a\| = k_{\mathbb{B}_2}^{(2)}(0, a) \leq k_D^{(2)}(0, a) \leq \tilde{k}_D(0, z_k) + \tilde{k}_D(z_k, a).$$

Letting $k \rightarrow \infty$ gives $k_D^{(2)}(0, a) = \tanh^{-1} \|a\|$. \square

Proof of Example 10. Setting

$$D = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_1^{34} z_2^{55}| < 1\},$$

then

$$c_D^*(0, z) = \max\{|z_1|, |z_1 z_2|, |z_1^2 z_2^3|, |z_1^5 z_2^8|, |z_1^{13} z_2^{21}|, |z_1^{34} z_2^{55}|\}$$

(see [1], Example 2.7.12). Since $\widehat{D} = \mathbb{D} \times \mathbb{C}$, we have that

$$c_D^*(0, z) > \widehat{h}_D(z) = |z_1| \iff |z_2| > 1.$$

On the other hand, note that $\{0\} \times \mathbb{C} \subset D$ and $\mathbb{D} \times \{z_2\} \subset D$, $|z_2| \leq 1$. So if $|z_1| < 1$, $|z_2| \leq 1$, then

$$k_D^{(2)}(0, (z_1, z_2)) \leq k_D(0, (0, z_2)) + k_D((0, z_2), (z_1, z_2)) \leq k_{\mathbb{D}}(0, z_1)$$

and hence

$$(k_D^{(2)}(0, (z_1, z_2)))^* \leq |z_1| = \widehat{h}_D((z_1, z_2)).$$

Since the opposite inequality always holds and ∂D contains a nontrivial analytic disc through any point $b = (b_1, b_2) \in \partial D$ with $|b_1| \neq |b_2|$, it follows that any point $a = (a_1, a_2)$ with $|a_1| < |a_2| = 1$ has the desired properties. \square

REFERENCES

- [1] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, de Gruyter Exp. Math. 9, de Gruyter, Berlin, New York, 1993.
- [2] M. Jarnicki, P. Pflug, *Extension of holomorphic functions*, de Gruyter Exp. Math. 34, de Gruyter, Berlin, New York, 2000.
- [3] N. Nikolov, P. Pflug, *Local vs. global hyperconvexity, tautness or k -completeness for unbounded open sets in \mathbb{C}^n* , Ann. Scuola Norm. Sup. Pisa Cl. Sci (5), 4, 601 – 618 (2005).
- [4] N. Nikolov, P. Pflug, *On the derivatives of the Lempert functions*, Ann. Mat. Pura Appl., DOI 10.1007/s10231-007-0056-z.

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